

Chapter 6 Read!

Properties of atoms

The Thomson model

The Rutherford model

Basic properties of atoms

1) Atoms are small!

e.g. Iron density 7.874 g/cm^3 ρ_{Fe}

molar mass $56.845(2)$ M_{Fe}

$N_A = 6.02214179(30) \times 10^{23}$

So, the volume occupied by a Fe atom is

$$\underbrace{\frac{M_{\text{Fe}}}{N_A}}_{\text{mass/Fe atom}} \frac{1}{\rho_{\text{Fe}}} = \frac{56.845 \text{ g}}{6.0221 \times 10^{23}} \frac{1}{7.874 \text{ g/cm}^3}$$

$$= 1.199 \times 10^{-23} \text{ cm}^3$$

$$r_{\text{radius}} = \left(\frac{3V}{4\pi} \right)^{1/3} = 0.1420 \text{ nm}$$

$$= 1.42 \text{ \AA (Angstrom)}$$

2) Atoms are (usually) stable

$\therefore$ There must be a balance of

a) attractive forces that keep it together

b) repulsive forces that keep it from collapsing

3) Atoms contain negatively charged electrons

JJ Thomson 1897, Nobel prize 1906

3 experiments 1) Cathode rays are made up of electric charge in such a way as to be inseparable magnetic field

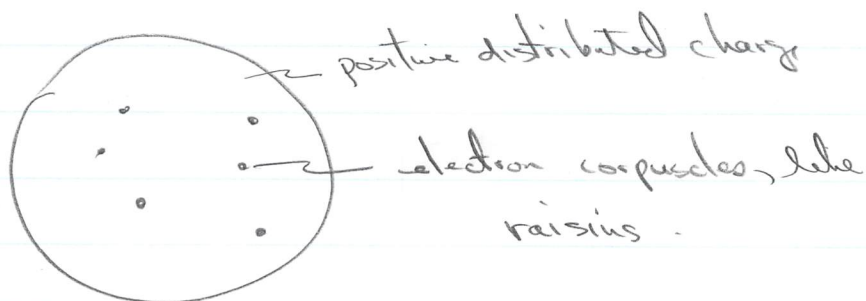
2) negative charge

3) e/m ratio m_e - thousand times less than H^+

corpuscles of charge

JJ Thomson's raisins in a pudding model

Thomson's model



These electrons, if perturbed, would oscillate.
Why?

Gauss's law

Aside:

$$\rho(\vec{r}) = \frac{Ze}{4\pi R^3} \left(\frac{r}{R}\right)^3 \quad r \leq R$$

$$= 0 \quad r > R$$

(1) $\Rightarrow$

$$E 4\pi r^2 \hat{r} = \frac{1}{\epsilon_0} \frac{1}{4\pi R^3} \frac{1}{3\pi} \frac{4\pi}{3} r^3$$

$$E \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{r}{R^3} \hat{r}$$

radial dir'n

$$\oint \vec{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int d\vec{x} \rho(\vec{x}) \quad (1)$$

↑
electric field vector

↑
charge density

$$\textcircled{2} \quad \vec{E} = \frac{Ze}{4\pi\epsilon_0} \frac{\vec{r}}{R^3} \quad \hat{r} \text{ radial direction vector}$$

$\uparrow$ radius of the atom

$$F = - \frac{Ze^2}{4\pi\epsilon_0 R^3} r \quad \textcircled{3} \quad \text{radial direction}$$

This is a spring, with spring constant

$$k = \frac{Ze^2}{4\pi\epsilon_0 R^3} \quad \textcircled{4}$$

and frequency

$$\omega^2 = \frac{Ze^2}{4\pi\epsilon_0 R^3 m_e}$$

not observed

$$\frac{k}{m} = \omega^2$$

⑤

oscillating charge would radiate at this frequency only

4) We know that when an atom is excited it radiates a whole bunch of frequencies. This is in contradiction to one of the consequences of Thomson's hypothesis. Therefore, the hypothesis was wrong!

We know, now in retrospect that the frequency spectrum is one of the components of the structure of a quantum system.

A bound system is characterized by a complete set of eigenfunctions

in 1D

$$U_n(x), E_n \quad n=0, 1, 2, \dots$$

The spectrum is given by the set

$$E_\gamma = E_m - E_n \quad m > n$$

But, at the time of Thomson, this information was confusing. (One brought clarity to this, (h.7))

Get another disproof of the Thomson model

Ernest Rutherford

1871-1937

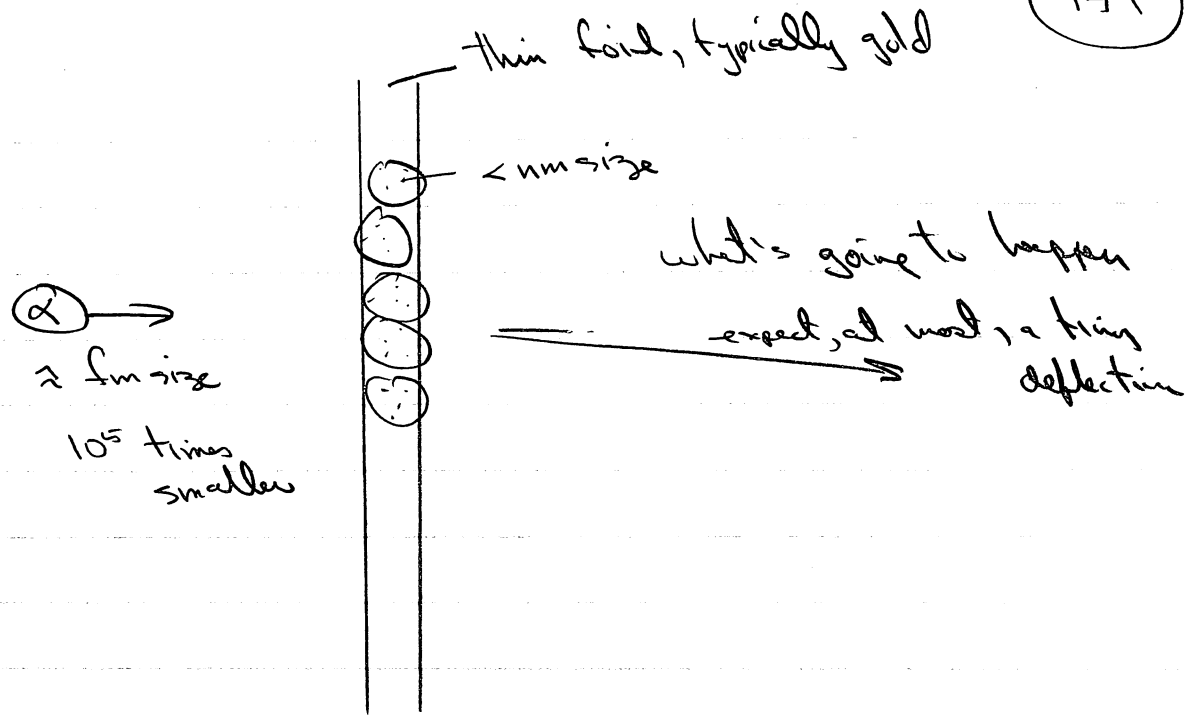
Nobel Prize in 1908

used beam of α particles

Thomson type experiments could be used to understand that

α 's have $2 \times$ the charge of e^- 's ; different sign

α 's outweigh e^- 's by 4000 times or so

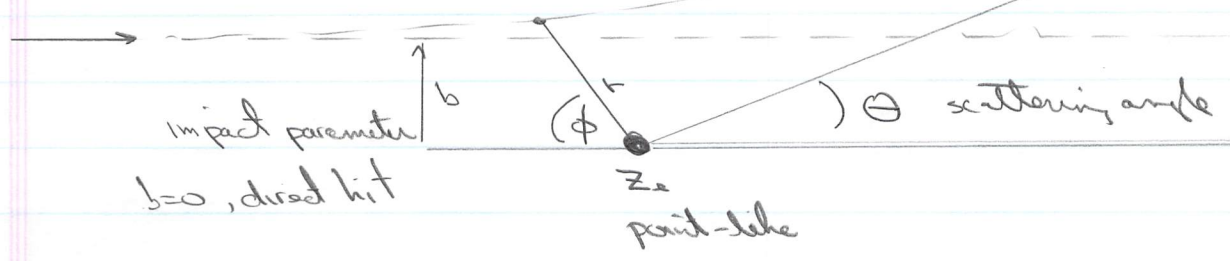


However, Ernest Rutherford and his junior colleagues, Ernest Marsden; Hans Geiger saw that 1 in 10^4 scattered backwards $> 90^\circ$

"like shooting a cannonball at tissue paper and having it bounce back"

Inescapable conclusion... the charge is gathered in the center, ^{is a "nucleus"} electrons must orbit the nucleus like planets orbit the sun.

Classical Rutherford scattering analysis



Classical mechanics predicts the trajectory
 # of charges of the projectile (Z in the case of α)

$$\frac{1}{r} = \frac{1}{b} \sin \phi + \frac{Z e^2 (\cos \phi - 1)}{8 \pi \epsilon_0 b^2 K} \quad (6)$$

First relate Θ , scattering angle to b , the impact parameter

kinetic energy of the projectile

Starting point $r \rightarrow \infty, \phi \rightarrow 0$

Final trajectory $r \rightarrow \infty, \phi \rightarrow \pi - \Theta$ (7)
 relate Θ to b

$$0 = \frac{1}{b} \sin(\pi - \Theta) + \frac{Z e^2 (\cos(\pi - \Theta) - 1)}{8 \pi \epsilon_0 b^2 K} \quad (8)$$

$$\sin(\pi - \Theta) = \sin \pi \cos \Theta - \cos \pi \sin \Theta$$

$$= \cos \pi \cos \Theta + \sin \pi \sin \Theta$$

$$0 = \sin \Theta + \frac{Z e^2 (\cos \Theta + 1)}{8 \pi \epsilon_0 b^2 K}$$

$$\frac{Z e^2 (\cos \Theta + 1)}{8 \pi \epsilon_0 b^2 K} = \sin \Theta$$

$$b = \left(\frac{32}{2K} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right) \left(\frac{\cos\theta + 1}{\sin\theta} \right) \quad (9)$$

$$\begin{aligned} \frac{\cos\theta + 1}{\sin\theta} &= \frac{\cos\left(\frac{\theta}{2} + \frac{\theta}{2}\right) + 1}{\sin\frac{\theta}{2} + \frac{\theta}{2}} = \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + 1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \frac{\cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2} \end{aligned}$$

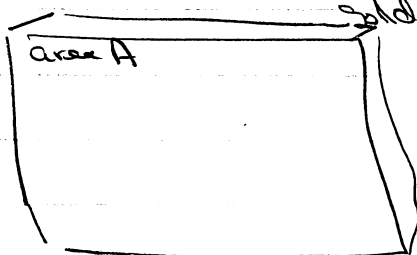
$$b = \left(\frac{32}{2K} \right) \frac{e^2}{4\pi\epsilon_0} \cot\frac{\theta}{2} \quad (10)$$

This tells us, measuring θ tells us how close it (the α) comes to the nucleus, b .

How can we use this?

Let's see what fraction of α 's backscatter

Thickness t



gold foil

density of atoms

$$n = \frac{\left(\frac{N_A \rho A t}{M} \right)}{\underbrace{A t}_{\text{volume of foil}}}$$

$$n = \frac{N_A \rho}{M}$$

Now, imagine that the foil is thin enough so that all particles are at the surface

all particles with $b < b_0$ will scatter backwards



$n t$ is the # of atoms / unit surface area

Fraction with $\theta > 90^\circ$, corresponds to

$$f_{\theta > 90^\circ} f_{b < b_0} = \underbrace{n t}_{\substack{\text{\# of} \\ \text{scattering} \\ \text{per unit} \\ \text{area}}} \underbrace{\pi b_0^2}_{\substack{\text{probability of scattering} \\ > 90^\circ}}$$

Gold foil $\rho = 19.3 \text{ g/cm}^3$ $M = 197 \text{ g/mole}$
 $t = 2 \times 10^{-4} \text{ cm}$ $K = 8 \text{ MeV}$ }
 typical values

$$b = \left(\frac{32}{2k} \right) \frac{e^2}{4\pi\epsilon_0} \cot \frac{\theta}{2}$$

 b_0

$$\theta_0 = \frac{\pi}{2}$$

$$f_{\theta > 90^\circ} = 7.5 \times 10^{-5}$$

$$f_{\theta > 45^\circ} = 3.6 \times 10^{-4}$$

The Rutherford Cross Section

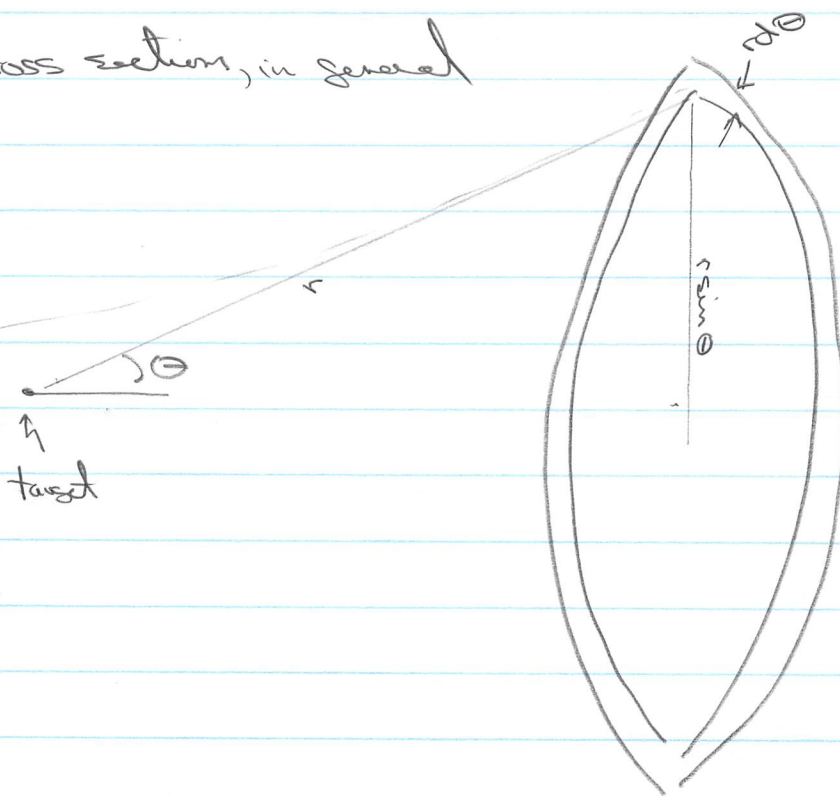
Classical Cross sections, in general

projectiles
→



↑
target

area of ring
 $b \rightarrow b+db$
 $= 2\pi b db$



area of ring $2\pi r \sin \theta r d\theta$
 $= 2\pi r^2 \sin \theta d\theta$

How many particles hit the detector?

We know that

$$f = nt \pi b^2$$

$$df = nt 2\pi b db$$

contribution to the fraction
from the ring

$$b = \left(\frac{Ze}{2K}\right) \left(\frac{e^2}{4\pi\epsilon_0}\right) \cot \frac{\theta}{2}$$

$$db = \left(\frac{Ze}{2K}\right) \frac{e^2}{4\pi\epsilon_0} \left(-\frac{1}{\sin^2 \frac{\theta}{2}}\right) \frac{1}{2} d\theta$$

$$\text{So } |df| = \left(\frac{Ze}{2K}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 nt \pi \frac{\cos \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}}$$

$$\frac{|df|}{dA} = \frac{\left(\frac{Ze}{2K}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 nt \pi \frac{\cos \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}}}{\underbrace{2\pi r^2 \sin \theta}_{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}}$$

$$\frac{|dS|}{dA} = \frac{(nt)}{4\pi r^2} \left(\frac{Ze}{2K}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

$\frac{dA}{4\pi r^2} =$ Fractional area of the sphere at
r that the particles from
 $b \rightarrow b+db$ go to
 $dA/4\pi r^2 = d\Omega$ solid angle

$|d\sigma|/nt =$ probability of deflection to angle $\Theta \rightarrow d\sigma$
 $d\Omega$

$dA/4\pi r^2 =$ fractional area of the spherical ring $\Theta \rightarrow d\sigma$
 $d\Omega$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{2K} \right)^2 \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{\sin^4 \frac{\Theta}{2}}$$

units: $\left(\frac{1}{E^2} \right) (E^2 L^2)$

systematics a) $\frac{d\sigma}{d\Omega} \propto Z^2$ ✓

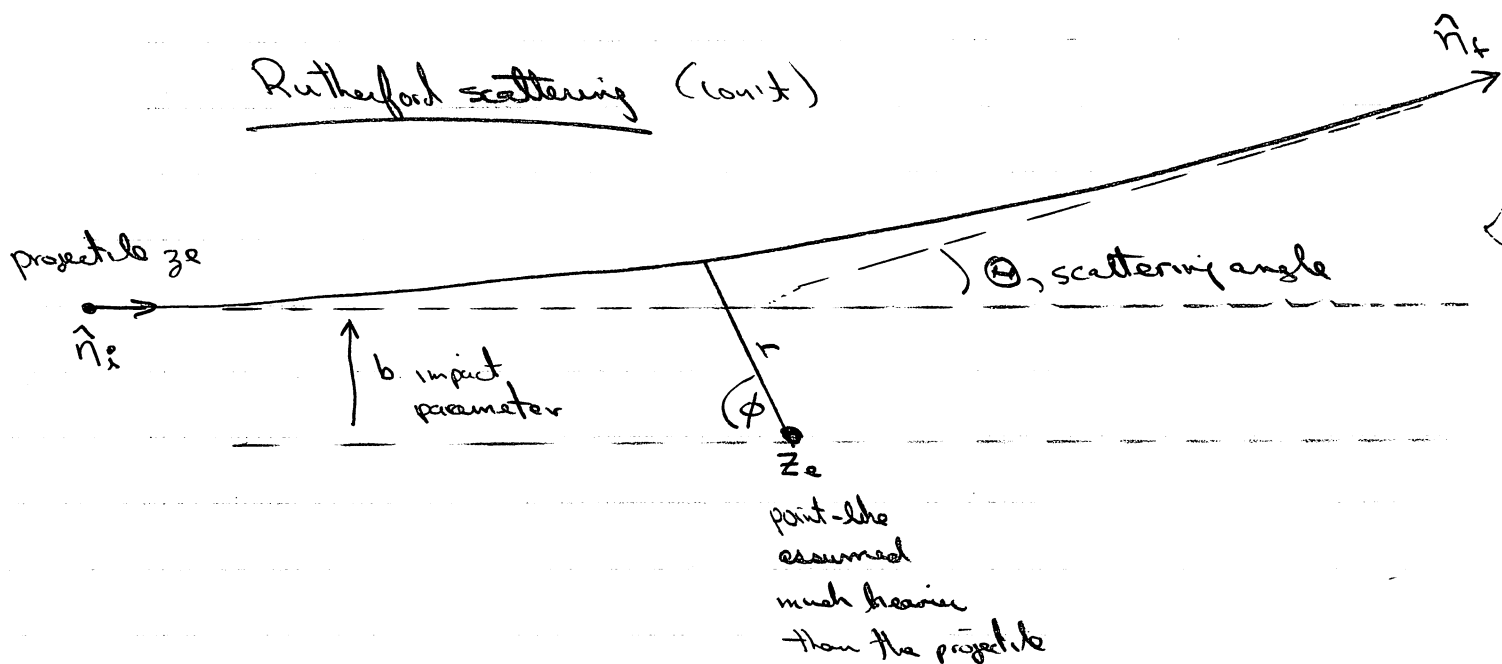
b) $\frac{d\sigma}{d\Omega} \propto \frac{1}{K^2}$ ✓

c) $\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4 \frac{\Theta}{2}}$ ✓

all based by Rutherford



Rutherford scattering (cont)



b & Θ are constants of the motion

r, ϕ are dynamic variables, that give the equation of motion.

From last time, r , the distance between the target & the projectile, is given by

$$\frac{1}{r} = \frac{\sin \phi}{b} + \left(\frac{z_e}{2K} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{1}{b^2} (\cos \phi - 1) \quad (K6.12) \quad (1)$$

The solution in the limit $r \rightarrow \infty$ has two solutions,

One is $\phi = 0$ when $r \rightarrow \infty$

The other can be used to solve for a special value of ϕ , we'll call, for the time being

Thus

$$\lim_{r \rightarrow \infty} \textcircled{1} \Rightarrow 0 = \frac{\sin \phi_0}{b} + \left(\frac{Ze}{2K} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{1}{b^2} (\cos \phi_0 - 1) \quad \textcircled{2}$$

One solution is $\phi_{0,i} = 0$, starting angle,
the other relates the final angle $\phi_{0,f}$ to b ,

$$b = \left(\frac{Ze}{2K} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{1 - \cos \phi_{0,f}}{\sin \phi_{0,f}} \quad \textcircled{3}$$

which can also be expressed in terms of the scattering angle

$$b = \left(\frac{Ze}{2K} \right) \left(\frac{e^2}{4\pi\epsilon_0} \right) \cot\left(\frac{1}{2}\Theta\right) \quad \textcircled{4} \quad (\text{K6.13})$$

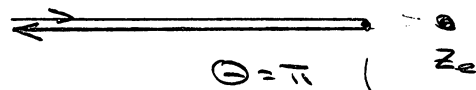
where $\Theta = \pi - \phi_{0,f} \quad \textcircled{5}$
(see figure)

To show $\phi_{0,f} = 2\left(\frac{\pi}{2} - \frac{\Theta}{2}\right)$
in $\textcircled{3}$ and use trig
identities

Distance of closest approach

2 approaches

Krone's Set $b=0$



at the turn around point, the particle stops, so all its energy is potential. Equating the initial kinetic energy K_i to this

$$K = \frac{Z_e^2}{4\pi\epsilon_0 r_{min}}$$

$$\text{Or } d = \left(\frac{Z_e^2}{K} \right) \frac{e^2}{4\pi\epsilon_0} \quad (5) \quad K6.22$$

Hence, we can rewrite (1) as

$$(5) \rightarrow (6) \Rightarrow \frac{1}{r} = \frac{\sin \theta}{b} + \frac{d}{2b^2} (\cos \theta - 1) \quad (6)$$

Another approach would be to find the minimum r for any input parameter, b . By symmetry, this occurs when $\phi_m = \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$ (or you can prove this, and find

$$r_{\min} = \frac{2b^2}{d} \frac{1}{\left[\sqrt{1 + \frac{4b^2}{d^2}} - 1\right]} \quad (7)$$

The $b \rightarrow 0$ limit is delicate because both numerator and denominator $\rightarrow 0$. Either using L'Hopital's rule, or Taylor expansions (below)

$$\begin{aligned} r_{\min} &\xrightarrow{b \rightarrow 0} \frac{\frac{2b^2}{d}}{1 + \frac{2b^2}{d^2} + O\left(\frac{b^4}{d^4}\right) - 1} \\ &= d, \text{ as expected} \end{aligned}$$

complete the square

or

$$\begin{aligned} r_{\min} &= \frac{2b^2}{d} \frac{\left[\sqrt{1 + \frac{4b^2}{d^2}} + 1\right]}{\frac{4b^2}{d^2}} \\ &= d \quad \checkmark \end{aligned}$$

The closest distance of approach is useful to determine the maximum energy of the projectile for which the Rutherford expression will still work, since the projectile will not have penetrated the target. Both are finite in size.

From 312, we require that (to be shown in 312)

$$d > 1.2 (A_p^{1/3} + A_t^{1/3}) \quad (\text{fm})$$

projectile target
atomic masses

eg. for α ($A=4$) on ^{197}Au ($A=197$)
 $Z=2$ $Z=79$

$$\frac{3Ze^2}{4\pi\epsilon_0 K} > 1.2 (A_p^{1/3} + A_t^{1/3})$$

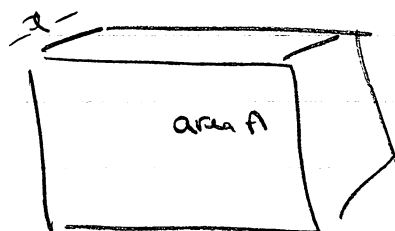
$$K < \frac{3Ze^2}{4\pi\epsilon_0} \frac{1}{1.2 (A_p^{1/3} + A_t^{1/3})}$$

$$< 25.6 \text{ MeV}$$

Analyzing Rutherford's experiment

Q. What fraction of the time did Geiger and Marsden observe a scatter in the back direction?

1) Find the density of scattering centers (# of nuclei / V in the target)



$$n = \frac{\overset{\text{Volume}}{N_A g(A t)} / (A t)}{M}$$

N_A Avogadro's number
atoms/mole $\sim 6.022 \times 10^{23}$

M = mass/mole ~ 56

g = density

$$n = \frac{N_A g}{M}$$

now choose a beam directed at this foil, with radius b , and choose b to give a 90° deflection.

b_0



α with $b = b_0$ 90°
 α with $b < b_0$ $> 90^\circ$

$$f_{\theta > \theta_0} = f_{b < b_0} = n \int \pi b_0^2$$

$\uparrow$ $\uparrow$
 density thickness of foil

recall that $b = \left(\frac{Ze}{2k} \right) \frac{e^2}{4\pi\epsilon_0} \cot \frac{\theta}{2}$

$$b_0 = \left(\frac{Ze}{2k} \right) \frac{e^2}{4\pi\epsilon_0} \cot \frac{\pi}{4}$$

for a foil 2×10^4 cm thick $K = 0.12$

$$f_{\theta > 90^\circ} = 7.5 \times 10^{-5}$$

$$f_{\theta > 45^\circ} = 3.6 \times 10^{-4}$$

$$f_{\theta > 10^\circ} = 0.0043 < 1/2\%$$

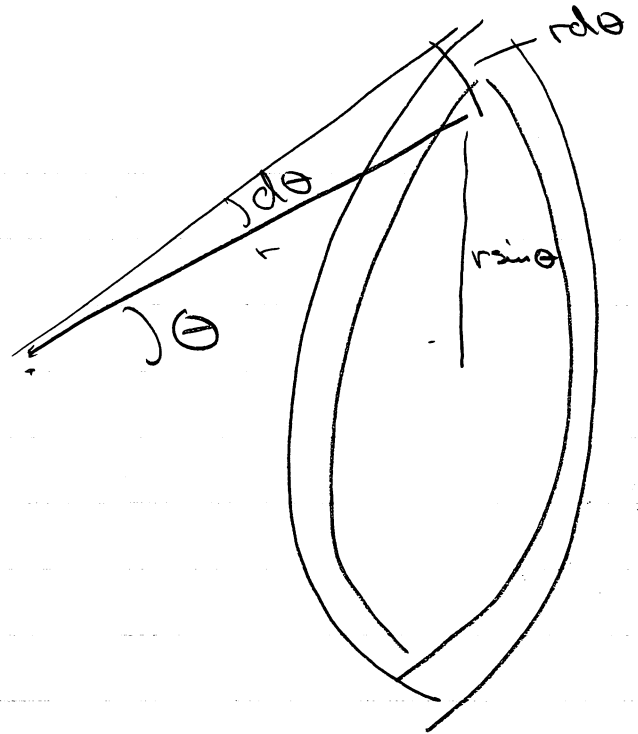
"very forward directed" scattering

How do we characterize the "directedness" of the scattering?

Differential cross section



ring source of
 α 's, all at the same
 impact parameter



$$b = \frac{3Z}{2K} \left(\frac{e^2}{4\pi\epsilon_0} \right) \cot \frac{1}{2}\theta \quad (8)$$

$$dA = 2\pi r \sin\theta r d\theta$$

So knowing b , we know the scattering angle

The $b \rightarrow b+db$ ring scatters into the $\theta \rightarrow \theta+d\theta$ ring
 radius of scattering $< b$

$$f_b = \pi b^2 n t \quad (\text{all impact parameters } < b)$$

$$df_b = 2\pi b db n t \quad (9)$$

(9) $\Rightarrow$ (show)

$$df = \left(\frac{3Z}{2K} \right)^2 \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 n t \pi \frac{\cos^2 \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}} d\theta$$

$$\frac{df}{dA} = \frac{(nt)}{4\pi r^2} \left(\frac{3Z}{2K} \right)^2 \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad (10)$$

$$\frac{[d\sigma/(nt)]}{[dA/4\pi r^2]} = \left(\frac{ze}{2K}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

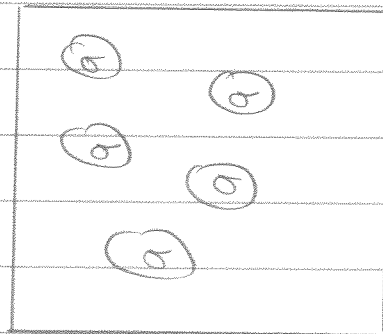
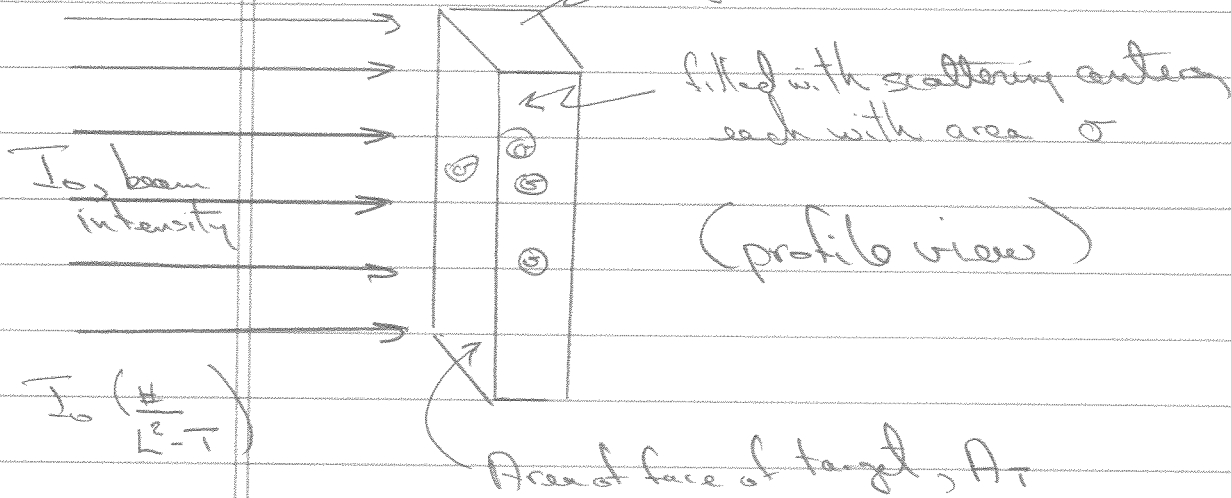
$$[d\sigma/(nt)] = \frac{\text{fraction that scatter}}{\text{that scatterers/unit area}} = d\sigma \uparrow$$

$$[dA/4\pi r^2] = d\Omega \quad \text{elemental solid angle}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{ze}{2K}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}}$$

Rutherford cross section

NERS 311

Cross Sections revisitedsubtle: How to understand the unseeable $t = \text{thickness}$
target

(beam's eye view)

Note: It is so thin that the scatterers do not overlap, and the particles in the beam interact with only one at a time.

N_s = number of scatters in the target

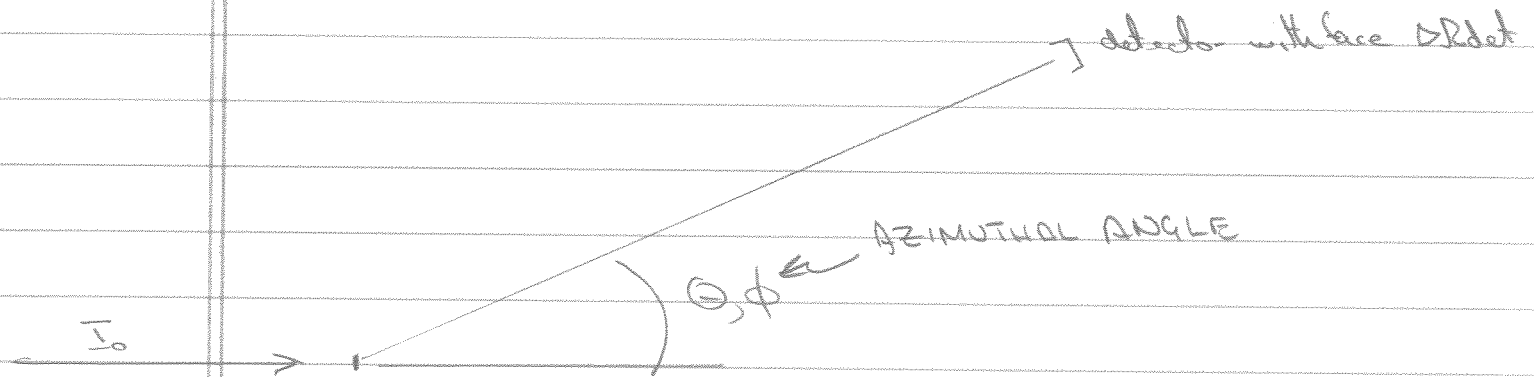
$$N_s = (A \cdot t) \left(\frac{N_A \rho_m}{M} \right)$$

Probability of interaction is $\frac{N_s \sigma}{A_T}$ } see 2nd figure previous page

Total number of particles that interact is

$$\underbrace{(I_0 A_T)}_{\substack{\text{\# in} \\ \text{beam} \\ \text{per unit} \\ \text{time}}} \underbrace{\left(\frac{N_s \sigma}{A_T} \right)}_{\substack{\text{prob of} \\ \text{interaction}}} = I_0 N_s \sigma$$

Zoom out on the experiment and place a detector



Detector measures $m(\Theta, \phi)$ events per unit time

The total number of scattered particles is /unit time)

$$\sum \Delta R_{det} m(\Theta, \phi)$$

sweep
detector
over all
space

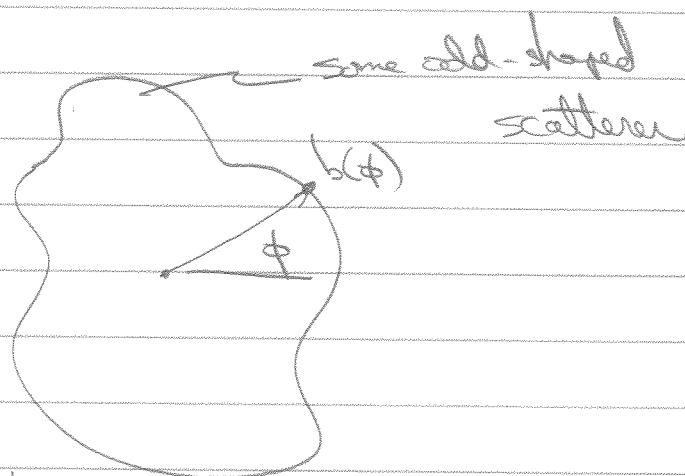
$$I_0 N_s \sigma = \sum_{\text{sum over all space}} \text{scat } m(\theta, \phi) = \underbrace{\left(\frac{I_0 A_t}{A_t} \right)}_{\text{has to be known}} \underbrace{\left(\frac{N_s}{A_t} \right)}_{\text{to this}} \underbrace{\sum_{\text{all space}} \text{scat } d\sigma(\theta, \phi)}_{\text{and this}} \underbrace{d\Omega}_{\sigma}$$

$$\text{So, } m(\theta, \phi) = \frac{I_0 N_s}{d\Omega} d\sigma(\theta, \phi)$$

$$\Rightarrow \text{has to get } \frac{d\sigma}{d\Omega} \text{ from } m(\theta, \phi)$$

If we could see the particles in the target
Classical picture

Beam's eye view



$$\sigma = \left| \int_0^{2\pi} d\phi \int_0^{b(\phi)} b db \right|$$

$$\text{So } d\sigma = |b db d\phi|$$

If you can not see the particle, but have a theoretical model for $b(\theta, d)$

e.g. Rutherford $b = \left(\frac{zZe^2}{2K} \right) \cot \frac{\theta}{2}$

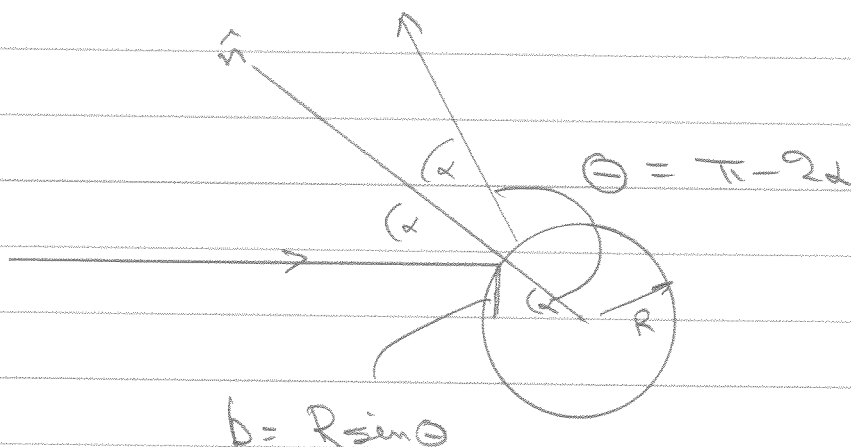
↖ does not depend on d

$$\frac{d\sigma}{d\Omega} = \frac{|b db d\phi|}{d\Omega} = \frac{|b db| d\phi}{\sin\theta d\theta d\phi}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{|b db|}{\sin\theta d\theta} = \left| \frac{b db}{d(\cos\theta)} \right|}$$

Can get Rutherford cross section for this

Another example, disco ball mirrors



$$\begin{aligned}
 b &= R \sin \alpha \\
 &= R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\
 &= R \cos \frac{\theta}{2}
 \end{aligned}$$

$$db = -\frac{R}{2} \sin \frac{\theta}{2} d\theta$$

$$\frac{|b db|}{\sin \theta d\theta} = \frac{R^2}{2} \frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\sin \theta}$$

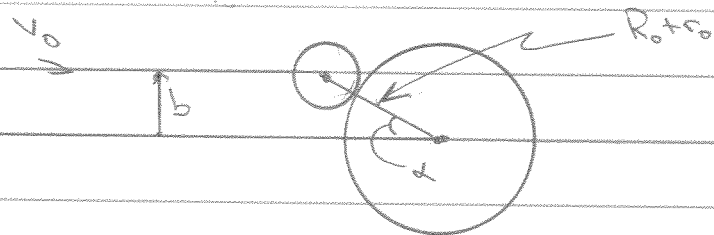
$$\frac{d\sigma}{dR} = \frac{R^2}{4}$$

$$\sigma = \int dR \frac{d\sigma}{dR} = \frac{R^2}{4} \int dR = \frac{R^2}{4} 4\pi$$

$$\sigma = \pi R^2, \text{ as expected}$$

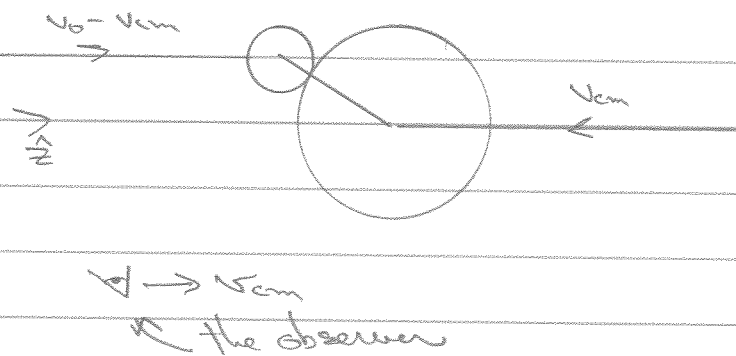
Note $\frac{d\sigma}{dR}$ does not depend on θ , i.e.
 $\Rightarrow$ scattering is isotropic

Next example small sphere mass m , radius r_0 from left
 large sphere mass M , radius R_0 , at rest



From the drawing, $b = (R_0 + r_0) \sin \alpha$

It is very difficult to understand how to solve this problem, unless we transform to the center of momentum frame

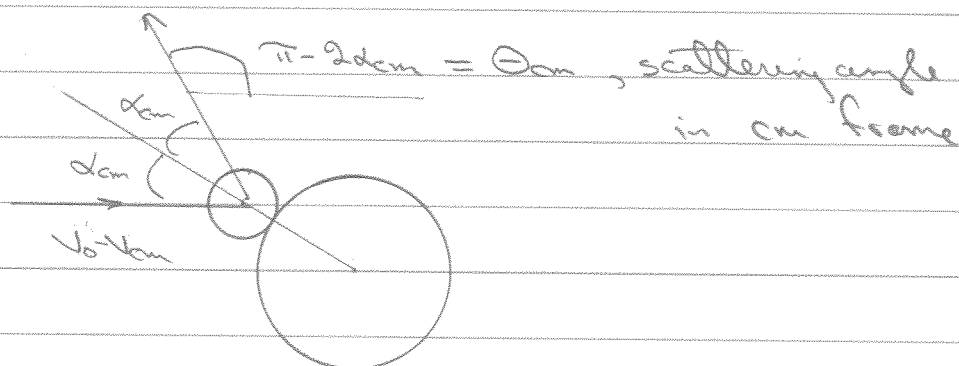


In the Center of Momentum (COM frame)

$$m(v_0 - v_{cm}) - Mv_{cm} = 0 \quad \text{COM} \cdot \hat{z}$$

$$v_{cm} = \frac{m v_0}{M + m}$$

In the CM frame



$$b = (R_0 + r_0) \sin \alpha_{cm}$$

$$= (R_0 + r_0) \sin \left(\frac{\pi}{2} - \frac{\theta_{cm}}{2} \right)$$

$$b = (R_0 + r_0) \cos \frac{\theta_{cm}}{2}$$

We have a relationship between b & θ_{cm} , so, we can get a cross section

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{|b db|}{\sin \theta_{cm}} = \frac{(R_0 + r_0) \cos \frac{\theta_{cm}}{2} (R_0 + r_0) \left(\frac{1}{2} \right) \sin \frac{\theta_{cm}}{2}}{\sin \theta_{cm}}$$

$$= \frac{(R_0 + r_0)^2}{2} \frac{\sin \frac{\theta_{cm}}{2} \cos \frac{\theta_{cm}}{2}}{\sin \theta_{cm}} = \frac{(R_0 + r_0)^2}{4}$$

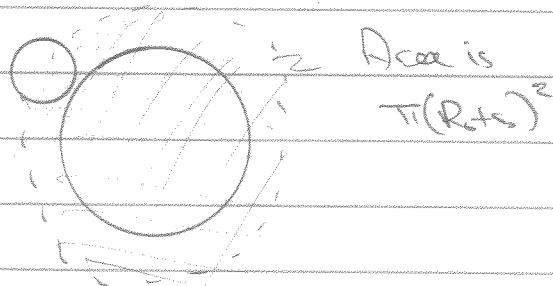
$$\boxed{\frac{d\sigma}{d\Omega_{cm}} = \frac{(R_0 + r_0)^2}{4}}$$

← no angular dependence, scattering is isotropic

Total cross section is

$$\sigma = \int dR_{ab} \frac{d\sigma}{dR_{ab}} = \pi(R_0 + r_0)^2$$

makes sense



Note that the cross section is independent of inertial frames, the differential cross section is not

In general

$$\frac{d\sigma}{dR_{ab}} = \frac{b(\theta_{ab}) db(\theta_{ab})}{\sin \theta_{ab}}$$

$\Rightarrow$

$$\left(\frac{d\sigma}{dR_{ab}} \right) = \left(\frac{d\sigma}{dR_{ab}} \right) \frac{\sin \theta_{ab} d\theta_{ab}}{\sin \theta_{ab} d\theta_{ab}}$$

It can be shown

$$\frac{\sin \theta_a d\theta_a}{\sin \theta_{ab} d\theta_{ab}} = \left\{ \frac{2r + \cos \theta_{ab} + [1 - r^2 \sin^2 \theta_{ab} + r^2 \cos^2 \theta_{ab}]}{[1 - r^2 \sin^2 \theta_{ab}]^{1/2}} \right\}$$

where $r = \frac{m}{M}$

One could use this for example, to calculate the effect on the cross section in the Rutherford case

$$\text{There } \frac{m}{M} = r \approx \frac{2}{A}$$

A target

[Computer demo]

Center of Mass Transformation in Classical Mechanics

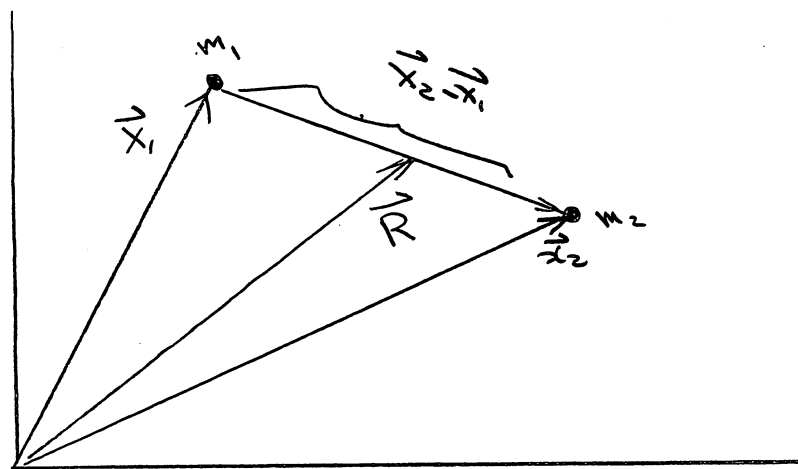
Consider a 2-body system with masses m_1 & m_2 undergoing a scattering via a potential that is central, i.e. $V(\vec{r}) = V(|\vec{r}_2 - \vec{r}_1|)$. That is, just the distance between the two objects is what the force depends upon.

The total energy of the system is

$$E_T = \frac{1}{2} m_1 \dot{\vec{x}}_1^2 + \frac{1}{2} m_2 \dot{\vec{x}}_2^2 + V(|\vec{x}_2 - \vec{x}_1|) \quad (1)$$

The position of the center of mass is

$$\vec{R} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \quad (2)$$



Let's rewrite $\vec{x}_1$ & $\vec{x}_2$ in terms of $\vec{R}$ and $\vec{r} = \vec{x}_2 - \vec{x}_1$.

$$(m_1 + m_2)\vec{R} = m_1\vec{x}_1 + m_2\vec{x}_2 \quad (2)$$

$$\vec{r} = \vec{x}_2 - \vec{x}_1 \quad (3)$$

$$(2) - m_2(3) \Rightarrow$$

$$(m_1 + m_2)\vec{R} - m_2\vec{r} = (m_1 + m_2)\vec{x}_1$$

$$\text{or } \vec{x}_1 = \vec{R} - \frac{m_2}{m_1 + m_2} \vec{r} \quad (4)$$

$$\text{Similarly } (2) + m_1(3) \Rightarrow$$

$$\vec{x}_2 = \vec{R} + \frac{m_1}{m_1 + m_2} \vec{r} \quad (5)$$

$$(4) \& (5) \rightarrow (1) \Rightarrow$$

$$E_T = \frac{1}{2}m_1 \left(\dot{\vec{R}} - \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \right)^2 + \frac{1}{2}m_2 \left(\dot{\vec{R}} + \frac{m_1}{m_1 + m_2} \dot{\vec{r}} \right)^2 + V(r)$$

or

$$E_T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + V(r)$$

$$M = m_1 + m_2 \quad \left\{ \begin{array}{l} \text{Total mass} \end{array} \right. \quad (6)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \left\{ \begin{array}{l} \text{"Effective" mass} \end{array} \right.$$

$V(r)$ only acts on $\vec{r}$, not $\vec{R}$, so

$\frac{1}{2}M\dot{\vec{R}}^2$ is the kinetic energy of the entire system, and is a constant of the motion

Let $K_{CM} = \frac{1}{2}M\dot{\vec{R}}^2$

$\frac{1}{2}M\dot{\vec{R}}^2 + V(r) = E_T - K_{CM}$
a constant

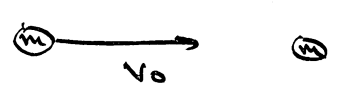
Note:

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

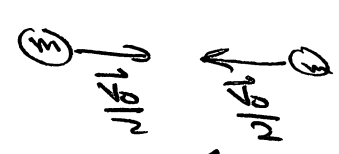
$\dot{\vec{R}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

If $\dot{\vec{R}} = 0$ $\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 = 0$
Zero momentum frame

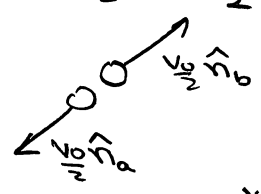
Application (1) billiard ball collision



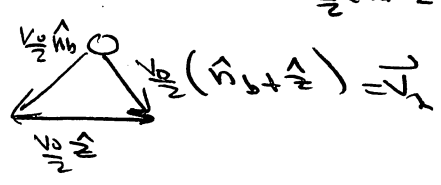
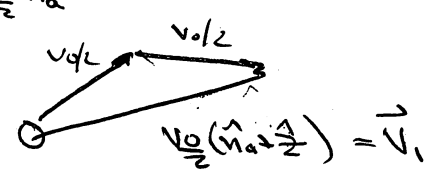
lab system



CM system



CM system



$\vec{V}_1 \cdot \vec{V}_2 = \left(\frac{v_0}{2}\right)^2 (\hat{n}_a + \hat{z})(\hat{n}_b + \hat{z})$

$= \left(\frac{v_0}{2}\right)^2 (\hat{n}_a \cdot \hat{n}_b + 2\hat{z} \cdot (\hat{n}_a + \hat{n}_b) + \hat{z} \cdot \hat{z})$

$$\vec{v}_1 \cdot \vec{v}_2 = 0$$

$$\Rightarrow \hat{n}_1 \cdot \hat{n}_2 \text{ are } \perp$$

Application (2)

Rutherford scattering

$$d\sigma = \left(\frac{Ze^2}{2K_{cm}} \right)^2 \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{4 \sin^4 \frac{\theta_{cm}}{2}} \sin \theta_{cm} d\theta_{cm} d\phi$$

is really derived in the CM frame.

What is $\frac{d\sigma}{d\Omega}$ / lab frame? (Use transformation of angle)

With this calculation, we can do
any Z, m_1 on any Z, m_2 .

Center of mass coordinates in a Quantum System

Consider the Schrödinger equation for two particles $m_1, \frac{1}{2}m_2$, interacting via a force that only depends on the distance between them:

$$\textcircled{1} \quad \left[-\frac{\hbar^2}{2m_1} \nabla_{\vec{x}_1}^2 - \frac{\hbar^2}{2m_2} \nabla_{\vec{x}_2}^2 + V(|\vec{x}_2 - \vec{x}_1|) \right] \psi(\vec{x}_1, \vec{x}_2) = E_T \psi(\vec{x}_1, \vec{x}_2)$$

E_T is the total system energy

Make a change of coordinates to center of mass and relative coordinates, that is,

$$\textcircled{2} \quad \vec{x}_1 = \vec{R} - \frac{m_2 \vec{r}}{M} ; \quad \vec{x}_2 = \vec{R} + \frac{m_1 \vec{r}}{M}$$

where $\vec{R} = (m_1 \vec{x}_1 + m_2 \vec{x}_2) / M$
 $\vec{r} = \vec{x}_2 - \vec{x}_1$
 $M = m_1 + m_2$

$$\textcircled{3} \quad \left\{ \begin{aligned} \nabla_{\vec{x}_1} &= \frac{\partial \vec{R}}{\partial \vec{x}_1} \frac{\partial}{\partial \vec{R}} + \frac{\partial \vec{r}}{\partial \vec{x}_1} \frac{\partial}{\partial \vec{r}} = -\frac{m_2}{M} \nabla_{\vec{R}} - \nabla_{\vec{r}} \\ \nabla_{\vec{x}_2} &= \frac{\partial \vec{R}}{\partial \vec{x}_2} \frac{\partial}{\partial \vec{R}} + \frac{\partial \vec{r}}{\partial \vec{x}_2} \frac{\partial}{\partial \vec{r}} = \frac{m_1}{M} \nabla_{\vec{R}} + \nabla_{\vec{r}} \end{aligned} \right.$$

③ $\Rightarrow$ ① $\Rightarrow$

$$\left\{ -\frac{\hbar^2}{2m_1} \left[\frac{m_1^2}{M^2} \nabla_{\vec{R}}^2 - \cancel{\frac{2m_1}{M} \vec{\nabla}_{\vec{R}} \cdot \vec{\nabla}_{\vec{r}}} + \nabla_{\vec{r}}^2 \right] \right.$$

$$\left. -\frac{\hbar^2}{2m_2} \left[\frac{m_2^2}{M^2} \nabla_{\vec{R}}^2 + \cancel{\frac{2m_2}{M} \vec{\nabla}_{\vec{R}} \cdot \vec{\nabla}_{\vec{r}}} + \nabla_{\vec{r}}^2 \right] \right.$$

$$\left. + V(r) \right\} \psi(\vec{R}, \vec{r}) = E \psi(\vec{R}, \vec{r})$$

④ or, $\left\{ -\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2 - \frac{\hbar^2}{2\mu} \nabla_{\vec{r}}^2 + V(r) \right\} \psi(\vec{R}, \vec{r}) = E \psi(\vec{R}, \vec{r})$
 where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

Since there is no dependence on $\vec{R}$ except in the $\psi(\vec{R}, \vec{r})$ wavefunction and $\nabla_{\vec{R}}^2$,

$-\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2$ is the KE operator on the total system

Thus $\psi(\vec{R}, \vec{r})$ separates into $\psi(\vec{R}) u(\vec{r})$

With $-\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2 \psi(\vec{R}) = E_{cm} \psi(\vec{R})$

④ becomes

$$-\frac{\hbar^2}{2\mu} \nabla_{\vec{r}}^2 u(\vec{r}) + V(r) u(\vec{r}) = \underbrace{(E - E_{cm})}_{\text{Call this } E} u(\vec{r})$$

So, we solve $u(\vec{r})$ but with the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$

Note that $\psi(\vec{r}) = A e^{i \vec{k} \cdot \vec{r}}$

describes a plane wave moving along a constant direction

$$k_{\text{cm}} = \left[\frac{2M K_{\text{cm}}}{\hbar^2} \right]^{1/2}$$

What a nice connection between classical and quantum physics!

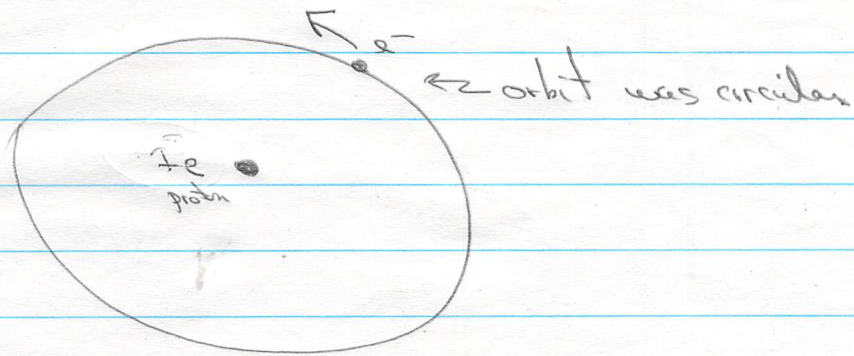
	Classical	Quantum
equation of motion (CM)	$E_{\text{cm}} = \frac{1}{2} M \dot{\vec{R}}^2$	$E_{\text{cm}} = -\frac{\hbar^2}{2M} \nabla_{\vec{R}}^2 \psi(\vec{R})$
interpretation	constant velocity one energy particle	fixed energy constant direction wave
equation of motion (relative)	$E = E_{\text{cm}} + K_{\text{rel}}$ $= \frac{1}{2} \mu \dot{\vec{r}}^2 + V(r)$	$E \psi(\vec{r})$ $= \left[-\frac{\hbar^2}{2\mu} \nabla_{\vec{r}}^2 + V(r) \right] \psi(\vec{r})$
interpretation	trajectory of particle with mass μ	wave equation for particle of mass μ

The Bohr Model of the Atom

- The Thomson Model had many difficulties associated with it (Nor 4 lecture notes)
- Rutherford discovered that the +ve charge of the atom was concentrated in a "point-like" object at the center (the nucleus) that also contained most of the mass. (99.975% of it, approximately)

Bohr's Model of the atom

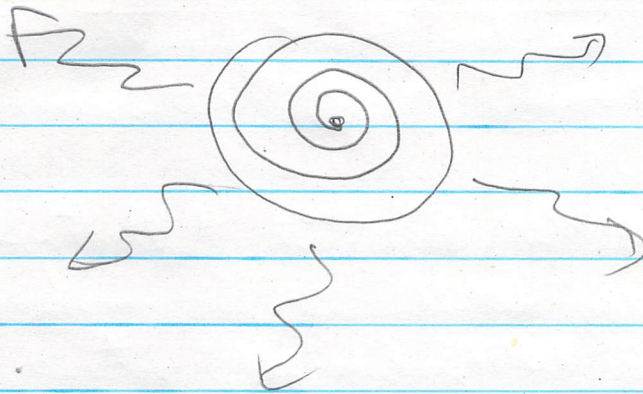
Ergo, the electron's must be in a planetary orbit about the nucleus. For the H-atom



Nice idea!

Problem: The electron in orbit experiences an acceleration (centrifugal force) that continuously bends the electron. Hence, it gives off synchrotron

radiation, and the atom collapses.



Power radiated
by a single
charge is

$$P = \frac{2}{3} \left(\frac{q^2}{4\pi\epsilon_0} \right) \frac{1}{c^3} |\vec{a}|^2$$

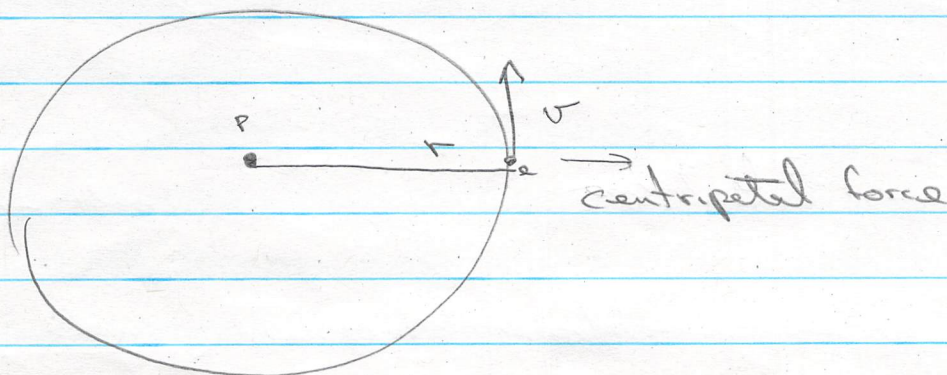
"acceleration"

(Classical Electrodynamics)

Bohr's ingenious planck

"Planck showed us that photons are 'quanta', hence the orbits must have fixed energy and photons result by means of a discrete jump in orbits downward in energy"

Follow this with some math:



The centrifugal force (pushing it out of orbit)

$$F_{\text{out}} = \frac{mv^2}{r} \quad (1)$$

must be balanced by the centripetal force (pulling it back in)
namely, the electrostatic force

$$F_{\text{in}} = \frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2} \quad (2)$$

Therefore $F_1 = F_2 =$

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2} \quad (3)$$

$$(3) \times \frac{r}{2} \Rightarrow \frac{1}{2}mv^2 = K = \frac{e^2}{8\pi\epsilon_0} \frac{1}{r} \quad (4)$$

The total energy $E = K + V$ ← potential

$$E = \frac{e^2}{8\pi\epsilon_0} \frac{1}{r} - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$

← from (4)
← from $E = \frac{1}{2}mv^2$

$$E = -\frac{e^2}{8\pi\epsilon_0} \frac{1}{r} \quad (5)$$

Now, for the genius moment

Bohr quantized the electron's angular momentum,
(because that's what gives rise to photons)

$$mV_n r_n = n\hbar \quad (6)$$

$\uparrow$
 quantum #

$$\therefore K_n = \frac{1}{2} m V_n^2 = \frac{1}{2} m \left(\frac{n\hbar}{m r_n} \right)^2 = \frac{e^2}{8\pi\epsilon_0} \frac{1}{r_n} \quad (7)$$

Solving for $\frac{1}{r_n}$ in (7)

$$\frac{1}{r_n} = \frac{m e^2}{4\pi\epsilon_0 \hbar^2} \frac{1}{n^2}$$

$\frac{1}{a_0}$ a_0 is Bohr's radius
(0.0529) nm
about $\frac{1}{2} \text{\AA}$

$$r_n = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} n^2 = a_0 n^2 \quad (8)$$

From (5) and (8)

$$E_n = -\frac{e^2}{8\pi\epsilon_0} \left(\frac{m e^2}{4\pi\epsilon_0 \hbar^2} \right) \frac{1}{n^2} \quad (9)$$

$-E_0 = -(13.6 \text{ eV})$

Another way that people who deal with higher energies like to write (9) is

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137.036} \dots \quad (10)$$

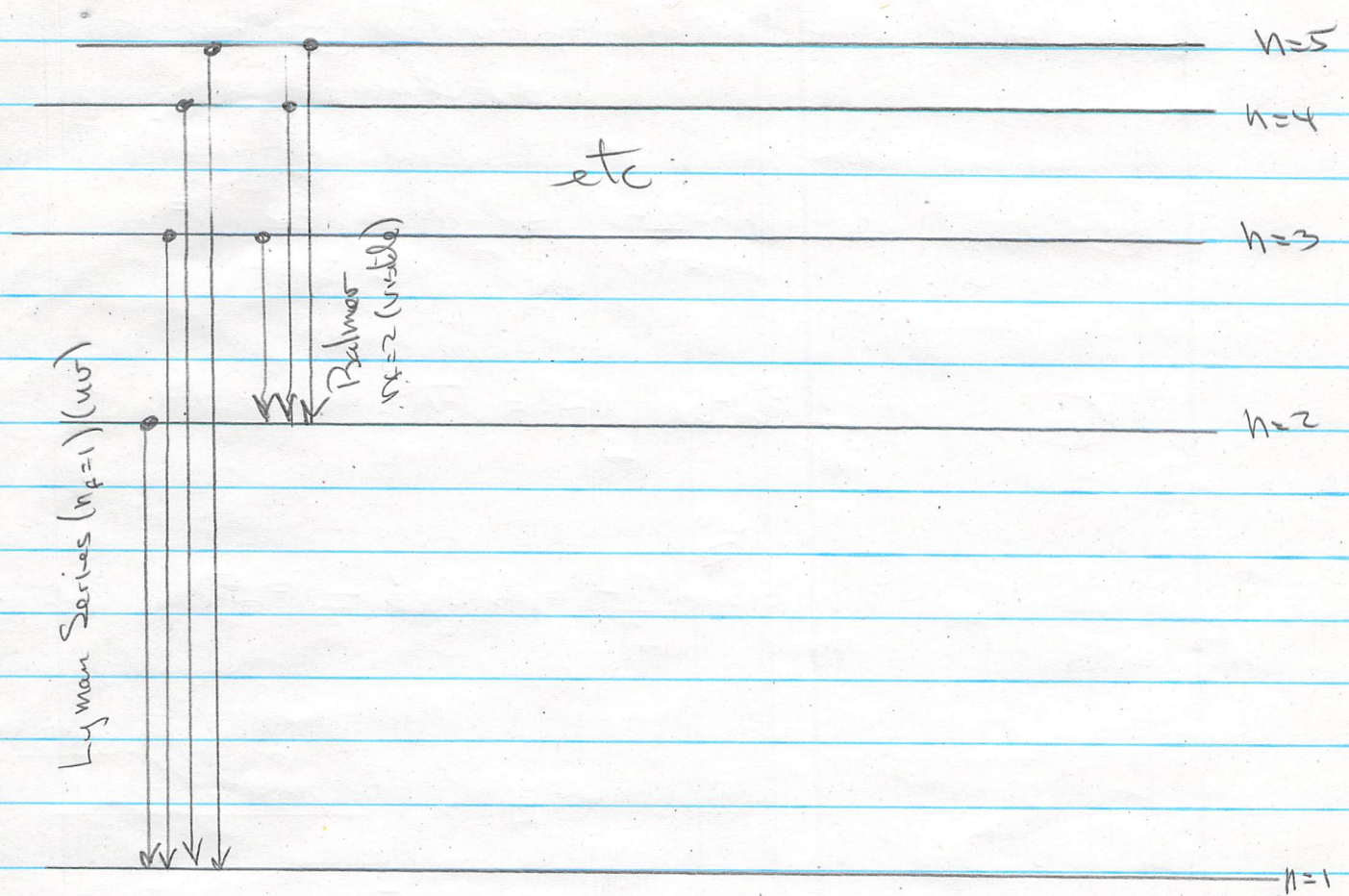
So that $E_n = -\frac{1}{2} m_e^2 \alpha^2 \frac{1}{n^2}$ (11)

$$E_0 = -\frac{1}{2} m_e^2 \alpha^2 = -13.6 \text{ eV}$$

Transitions?

$$E_x = -E_0 \left\{ \frac{1}{n_f^2} - \frac{1}{n_i^2} \right\}$$

photon energy $E_x = E_0 \left\{ \frac{1}{n_f^2} - \frac{1}{n_i^2} \right\}$ (12)



$n_f = 1$	Lyman series (uv)
$n_f = 2$	Balmer " (visible)
$n_f = 3$	Paschen " (near ir)
$n_f = 4$	Brackett " (ir)
$n_f = 5$	Pfund " (far ir)
$n_f = 6$	Humphreys
$n_f = 7$	} Traditional & Anonymous
$n_f = 8$	

Fantastic experimental agreement with the Bohr model.

Frank-Hertz experiment

- demonstrated the existence of discrete energy levels

Correspondence Principle

- shows that at large quantum numbers, n , in the Bohr model, the classical radiation frequency is observed.

Deficiencies of the Bohr model (1913)

- a) - neglected finite mass

$$E_n = -E_0 \frac{1}{n^2}$$

$$E_0 = \left(\underbrace{\frac{e^2}{4\pi\epsilon_0\hbar c}}_{\alpha = \frac{1}{137}} \right)^2 \frac{1}{2} m_e c^2$$

(easy fix)

$$E_0 = \frac{\alpha^2 m_e c^2}{2} = 13.6 \text{ eV}$$

With finite mass

$$m_e \rightarrow \frac{m_e m_p}{m_e + m_p} = \mu$$

decreases E_0 by $\frac{1}{1.00055} \Rightarrow$ increases λ wavelength by 1.00055

- b) Measured wavelengths not measured in vacuum, but in air, $\Rightarrow$ Decrease λ wavelength

(easy fix)

by 1.00029, partially undoing the effect in a).

- c) The transition lines are actually two very closely spaced lines, no prediction from Bohr theory. (Fine structure, due to interaction of the electron spin with the magnetic field generated by its orbit) Spin proposed in 1925 by Goudsmit & Uhlenbeck

- d) Bohr theory $\Rightarrow$ $n=1$ ground state predicts that it has one unit of angular momentum. Experiments have determined that it is 0.

- e) violates HUP $\Delta p \Delta r \geq \hbar/2$ Heisenberg 1927
 $\uparrow$
 r can't be known exactly

$\Rightarrow$ we need do better

1926 Schrödinger introduces wave mechanics (quantum mechanics)